Deep Patterns in Algebraic Structures: Computational Approaches to Quiver Mutation and Exceptional Sequence Classification

J. E. Gordon • R. Z. Maresca
Mathematics Department, Bowdoin College, Brunswick, Maine

Exceptional Sequences

Let Δ be a quiver with no loops or cycles (e.g., A_n: $1 \to 2 \to \cdots \to n$)
A **complete exceptional sequence** in mod-$\Bbbk Q$ is an ordered tuple $(E_1, \cdots, E_n)$ of indecomposable modules satisfying:

- $\mathrm{Hom}(E_i, E_i) = \Bbbk$ and $\mathrm{Ext}^1(E_i, E_j) = 0$
- $\mathrm{Hom}(E_i, E_j) = 0 = \mathrm{Ext}^1(E_i, E_j)$ for $i > j$
- n is the number of vertices in the quiver

Why count them? Exceptional sequences connect to:

- Mutations and braid group actions
- Cluster algebras and Coxeter combinatorics
- Noncrossing partitions and parking functions

Known Count [Obaid-Nauman-Al Shammakh-Fakih-Ringel, 2013]:

$$E(\Delta) = \frac{n!\, h(\Delta)^n}{|W(\Delta)|}$$

W (Δ) is the Weyl group of type Δ
h(Δ) the corresponding Coxeter number.

Our question: Can we refine this count with Hom-Ext isoclasses?

The Hom-Ext Quiver

Given an exceptional collection $\{E_1, \cdots, E_n\}$, the **Hom-Ext** quiver (B,Z) is the quiver of the Ext Algebra [Igusa-Maresca, 2025]:

- Vertices: the modules $E_1, \cdots, E_n$
- Arrows: irreducible morphisms and extensions, mod factorization
- Relations Z: linear combinations of paths that compose to zero

Igusa and Maresca [Thrm 3.4]:

$$\#\{\text{exceptional orderings}\} = \mathrm{LinExt}(B)$$

Two exceptional collections are Hom–Ext isomorphic if their quivers with relations (B, Z) are isomorphic. Key quantities for each isoclass:

- LinExt(B) = number of linear extensions of poset B
- |Aut(B, Z)| = automorphisms preserving B and Z
- m(B,Z) = the number of τ-shifts mod h that preserve the Hom–Ext isoclass; how many different derived placements of the same Hom–Ext shape exist.

Hom-Ext Isoclass Formula

Conjecture 1: For Dynkin quiver Δ with Coxeter number h(Δ),

$$E(\Delta) = h(\Delta) \sum_{[B,Z]} \frac{\mathrm{LinExt}(B) \cdot m(B,Z)}{|\mathrm{Aut}(B,Z)|}$$

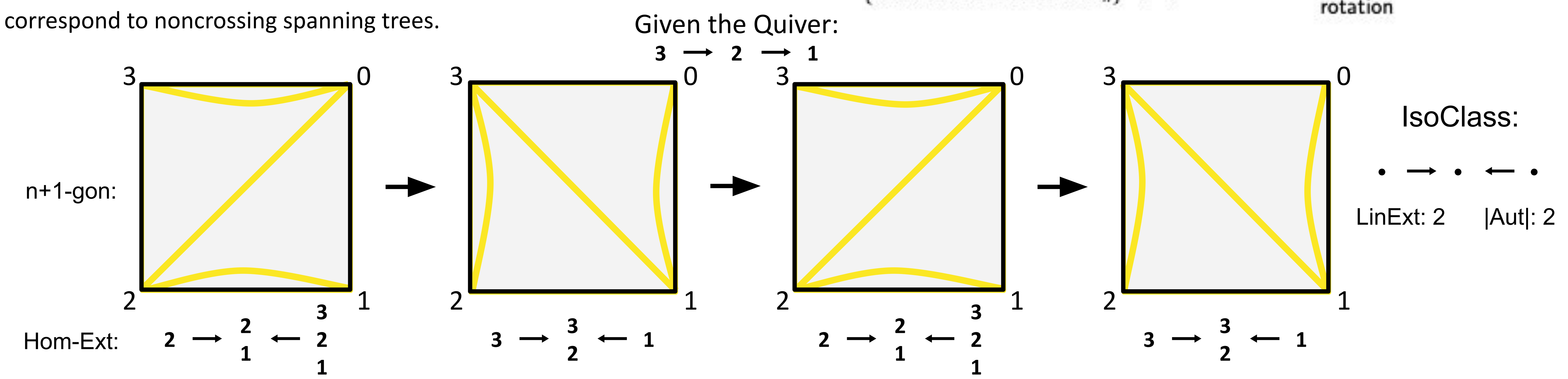
summing over Hom-Ext isoclasses

Geometric Model for Hom-Ext Isoclasses in Type A_n

Geometric realization from [Araya, 2011]: Indecomposable type A_n modules correspond to chords on an (n+1)-gon. Exceptional collections correspond to noncrossing spanning trees.

Conjecture 2:

$$\{\text{Hom–Ext isoclasses in } A_n\} \longleftrightarrow \frac{\{\text{Noncrossing trees on } (n+1)\text{-gon}\}}{\text{rotation}}$$



The Coxeter element (Tau) acts by rotation of the polygon. A single Hom-Ext isoclass corresponds to an orbit of noncrossing trees under this rotation

Counting Exceptional Sequences with Degree of Compatibility Graph

Build the Hom-Ext compatibility graph:

- Vertices: indecomposables in mod-$\Bbbk Q_n$
- Edges: $X_u \to X_v$ if $\mathrm{Hom}(X_u, X_v) = 0 = \mathrm{Ext}^1(X_u, X_v)$

For each indecomposable u:

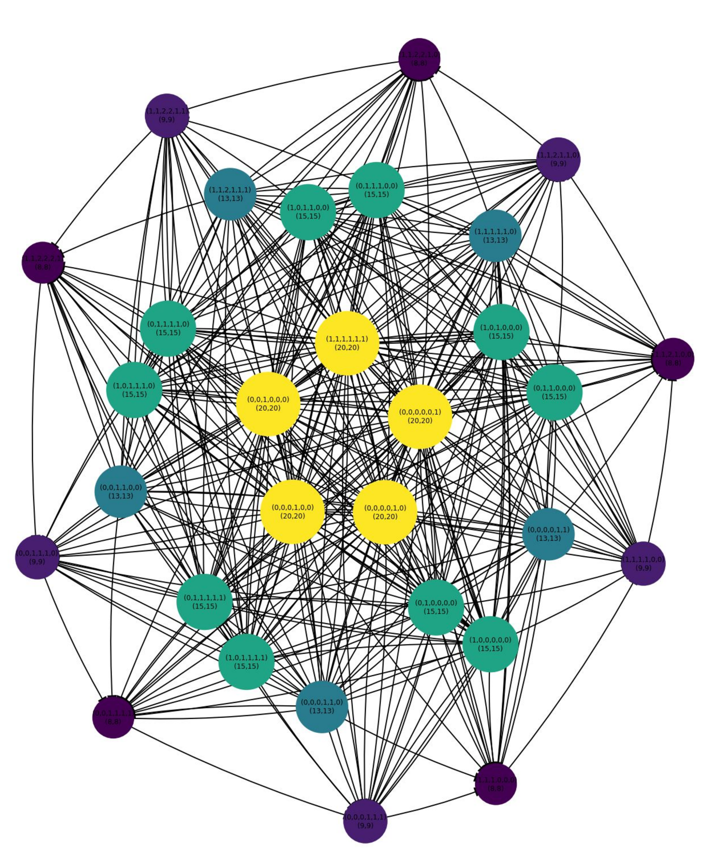
let $E(u) = \#\{\text{exc. seq. starting with } X_u\}$

Key Discovery: E(u) is constant in each shell!

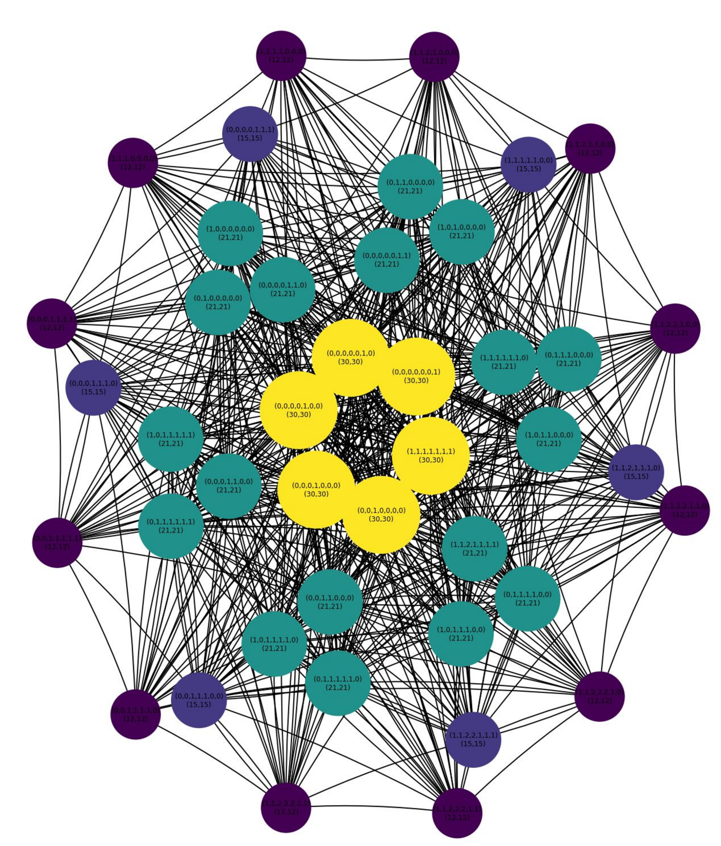
- $\deg^{out/in}(u)$ mostly explains E(u)
- Heat Kernel Signature captures finer splitting

Conjecture 3: E(u) can be expressed as a formula in degree and diffusion invariants

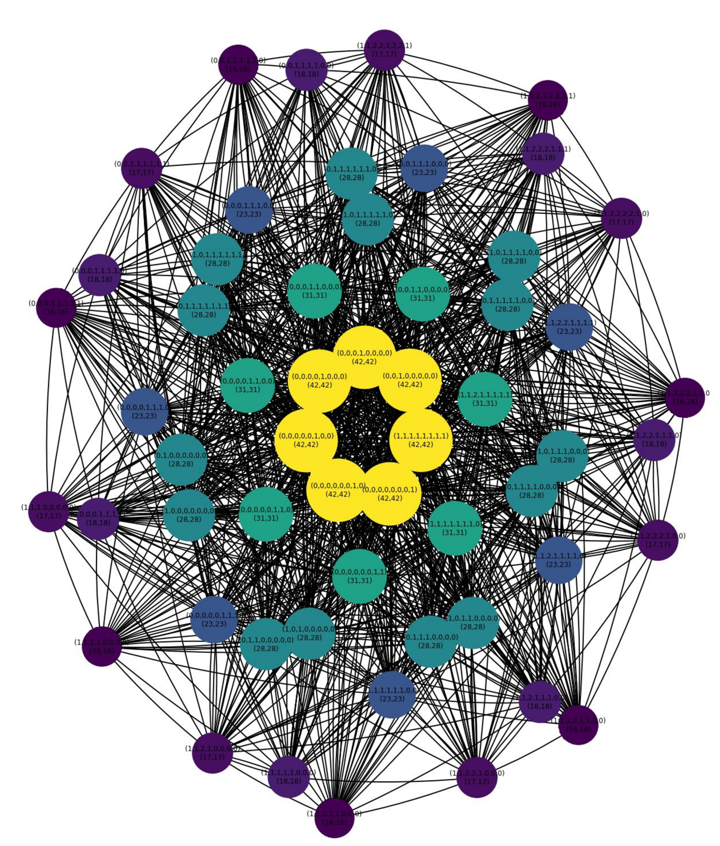
This would be a coarser decomposition than in [Obaid-Nauman-Al Shammakh-Fakih-Ringel, 2013]



Force Directed Compatibility Graph for D_6



Force Directed Compatibility Graph for D_7



Force Directed Compatibility Graph for D_8

Tilting Theory and Derived Equivalences

From [Assem-Happel, 1981],

- A tilting module $T = \oplus E_i$ determines an exceptional collection $\{E_1, \cdots, E_n\}$
- The algebra End(T) is a **tilted algebra** derived equivalent to mod-$\Bbbk\Delta$
- Pick a tilting module T' in mod-End(T), then $\mathrm{End}_{\mathrm{End}(T)}(T')$ is an **iterated tilted algebra** that is also derived equivalent to mod-$\Bbbk\Delta$

Conjecture 4: In Dynkin type A_n, the iterated tilted algebras (up to isomorphism) are in bijection with Hom-Ext isoclasses of exceptional sets

Two exceptional collections are Hom-Ext isomorphic if and only if they differ by a derived autoequivalence.

Future Work

- Verify Conjectures 1-4
- Closed formula using the compatibility graph
- Geometric model for Type D_n, possibly using a spoked polygon model with 2(n−1) vertices
- Extend work to Euclidean types

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